

Capacity of a condenser whose plates are circular arcs

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Abstract. We find an asymptotic formula for the conformal capacity of a plane condenser both plate of which are concentric circular arcs as the distance between them vanishes. This result generalizes the formula for the capacity of parallel linear plate condenser found by Simonenko and Chekulaeva in 1972 and sheds light on the problem of finding an asymptotic formula for the capacity of condenser whose plates are arbitrary parallel curves. This problem was posed and partially solved by R. Kühnau in 1998.

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1. Introduction. A pair of closed non-empty and non-intersecting subsets E_0, E_1 of the extended complex plane $\overline{\mathbb{C}}$ will be called *a condenser* and denoted by $C = (E_0, E_1)$. The sets E_0, E_1 are called *the plates* and the open set $G = \overline{\mathbb{C}} \setminus (E_0 \cup E_1)$ is called *the field* of the condenser C . *Capacity* of C is defined by

$$\text{Cap}(C) = \inf_{u \in \text{adm}(C)} \int_{\mathbb{C}} |\nabla u|^2 d\sigma(z), \quad (1)$$

where integration is with respect to the planar Lebesgue measure, and $\text{adm}(C)$ denotes the collection of continuous functions $\overline{\mathbb{C}} \rightarrow \mathbf{R}$ satisfying Lipschitz condition in a neighborhood of every finite point in G , possibly excluding a finite number of points, and such that $u(z) = i$ for $z \in E_i, i = 0, 1$. The function ω which is harmonic in G and assumes the value i on E_i is called the *potential function* of the condenser C . According to the Dirichlet principle it solves the extremal problem (1). The potential function exists for all condensers encountered in this paper. The lines orthogonal to the level curves of ω (or equivalently the level curves of the harmonic conjugate of ω) are called *the field lines* of the condenser C . The crucial property of the family Γ comprising the field lines of C is that its modulus $M(\Gamma)$ (the number reciprocal to the extremal length, see [1]) is equal to the capacity of C [8, 15]:

$$M(\Gamma) = \text{Cap}(C). \quad (2)$$

Both quantities in the above equality are conformal invariants. For the purposes of this paper it is sufficient to mention that the modulus of the family of straight line segments connecting the sides of length a of a rectangle and going parallel to the side of length b equals a/b [1].

Condensers defined above and their various generalizations found numerous applications in geometric theory of functions and many other areas. The study of the asymptotic behavior of the capacity of a specifically designed condenser led V.N. Dubinin to the definition and calculation of the generalized reduced modulus particular cases of which go back to Grötzsch, Teichmüller, Ahlfors and Beurling. Using this concept Dubinin and his students proved a number of covering and distortion theorems for analytic functions and gave solutions to several previously unsolved

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extremal partition problems. Many new inequalities for polynomials and rational functions have been also established. See for instance [5] and references therein.

The condenser $C_0 = (E_0, E_1)$ whose plates are parallel linear segments

$$E_0 = \{z : z \in [-L/2, L/2]\}, \quad E_1 = \{z : z \in [-L/2 + ih, L/2 + ih]\},$$

was studied by Simonenko and Chekulaeva in [13]. This condenser may also be thought of as a three dimensional condenser consisting of two parallel infinite bands. The authors used the following transcendental equation for the capacity of C_0 (which is essentially contained already in [3]):

$$\frac{\pi L}{2h} = KE(\phi, k) - EF(\phi, k), \quad (3)$$

where

$$\phi = \arcsin\left(\frac{1}{k}\sqrt{1 - E/K}\right) \quad \text{and} \quad \text{Cap}(C_0) = K/K'.$$

Here $K = K(k)$ is the complete elliptic integrals of the first kind (see (37)), $K' = K(\sqrt{1 - k^2})$, $E = E(k)$ is the complete elliptic integral of the second kind and $F(\phi, k)$, $E(\phi, k)$ are the incomplete elliptic integrals of the first and second kind, respectively (see (46), (45)). From this equation Simonenko and Chekulaeva derived an asymptotic formula for the capacity of C_0 when plates approach each other (i.e. $h \rightarrow 0$) which in our notation can be written as:

$$\begin{aligned} \text{Cap}(C_0) = & \frac{L}{h} + \frac{1}{\pi} \ln \frac{1}{h} + \frac{1}{\pi} (1 + \ln(2\pi L)) + \frac{h}{\pi^2 L} \ln \frac{1}{h} + \frac{h}{\pi^2 L} \left(\frac{1}{2} + \ln(2\pi L) \right) \\ & - \frac{h^2}{2\pi^3 L^2} \ln^2 \frac{1}{h} + \frac{h^2}{\pi^3 L^2} \left(\frac{1}{2} - \ln(2\pi L) \right) \ln \frac{1}{h} + \mathcal{O}(h^2). \end{aligned} \quad (4)$$

Here and below $f(x) = \mathcal{O}(g(x))$ as $x \rightarrow a$ means that $C_1 g(x) \leq |f(x)| \leq C_2 g(x)$ for all x in a neighbourhood of a .

Reiner Kühnau considered in [11] a general parallel plate condenser $C_K = (E_0, E_1)$ whose plates are arbitrary parallel curves. More precisely, let a Jordan curve parameterized by the arc length $s \in [0, L]$ be defined by a three times differentiable function $z = z(s)$. The plates E_0 , E_1 of the condenser C_K are curves parallel to $z(s)$ and located at equal distances from it:

$$E_0 = \{z : z_+(s) = z(s) + ihz'(s)/2, \quad s \in [0, L]\}, \quad E_1 = \{z : z_-(s) = z(s) - ihz'(s)/2, \quad s \in [0, L]\}.$$

Using a properly chosen family of curves Kühnau has proved the asymptotic formula

$$\text{Cap}(C_K) = \frac{L}{h} + \frac{1}{\pi} \ln \frac{1}{h} + \mathcal{O}(1). \quad (5)$$

The goal of this paper is to find an asymptotic expansion for the capacity of the condenser C_ρ which is a particular case of Kühnau's condenser when $z(s)$ traverses a circular arc symmetric with respect to real axis:

$$z(s) = \rho \exp[i(s/\rho - \gamma)], \quad s \in [0, L], \quad L = 2\gamma\rho.$$

Differentiation yields:

$$z_\pm(s) = (\rho \mp h/2) \exp[i(s/\rho - \gamma)]. \quad (6)$$

Hence the plates of the condenser $C_\rho = (E_0, E_1)$ are

$$E_0 = \{z : z = z_+(s), s \in [0, L]\}, \quad E_1 = \{z : z = z_-(s), s \in [0, L]\}. \quad (7)$$

Kühnau's approximation for its capacity is given by (5). The main result of this paper is

Theorem 1 For $h \rightarrow 0$ the following asymptotic expansion holds true

$$\begin{aligned} \text{Cap}(C_\rho) = & \frac{L}{h} + \frac{1}{\pi} \ln \frac{1}{h} + \frac{1}{\pi} (1 + \ln(4\pi\rho \sin(L/2\rho))) + \frac{\cot(L/2\rho)}{2\pi^2\rho} h \ln(1/h) + \frac{\cot(L/2\rho)}{2\pi^2\rho} h \times \\ & \times \left[\frac{1}{2} + \ln(4\pi\rho \sin(L/2\rho)) \right] - \frac{h^2 \ln^2(1/h)}{8\pi^3\rho^2 \sin^2(L/2\rho)} - \frac{h^2 \ln(1/h) (2 \ln(4\pi\rho \sin(L/2\rho)) - \cos(L/\rho))}{8\pi^3\rho^2 \sin^2(L/2\rho)} + \mathcal{O}(h^2). \end{aligned} \quad (8)$$

Our method allows one to compute as many terms in the above asymptotic expansion as one wishes. Formula (4) for the capacity of parallel linear plate condenser follows from (8) if we let $\rho \rightarrow \infty$. We have also obtained an analogue of equation (3) - see Theorem 2. Formula (51) from this Theorem has been announced in [6]. An equation for the capacity of a condenser one plate of which is a circular arc and the other is a radius of the same circle was found in [4].

Related developments in three dimensions were considered in [10] and [14]. The latter paper gives an asymptotic expansion for the capacity of a condenser whose plates are parallel planar figures of arbitrary shape as the distance between plates vanishes. Specific calculation of coefficients, however, has only been done in the classical case of the circular plates. In [10] R. Kühnau gives an asymptotic formula for the capacity of a condenser of which both plates are parallel finite surfaces when its field is restricted to the space between them.

2. Preliminaries. To derive a formula for the capacity of C_ρ it is sufficient to find one for C_1 . Once this has been done the conformal mapping $z \rightarrow \rho z$ will bring the result for C_ρ . Thus we take $\rho = 1$, $L = 2\gamma$ and $\delta = h/2$. Before proceeding further we transform our condenser C_1 into the condenser C'_1 which will be more convenient to work with. To this end carry out the conformal mapping

$$z \rightarrow \frac{z}{\sqrt{1-\delta^2}}.$$

Under this mapping the condenser C_1 transforms into condenser C'_1 with plates

$$E_0 = \left\{ z : z = e^{i\phi}/R, R = \sqrt{\frac{1+\delta}{1-\delta}}, \phi \in [-\gamma, \gamma] \right\}, \quad (9)$$

$$E_1 = \left\{ z : z = Re^{i\phi}, R = \sqrt{\frac{1+\delta}{1-\delta}}, \phi \in [-\gamma, \gamma] \right\} \quad (10)$$

(the new variable has been again denoted by z). Put $\varepsilon = R - 1$. Then:

$$\varepsilon = R - 1 = \sqrt{\frac{1+\delta}{1-\delta}} - 1 = \delta \left(1 + \frac{\delta}{2} + \frac{\delta^2}{2} \right) + \mathcal{O}(\delta^4) \quad (11)$$

and

$$\delta = \frac{(1+\varepsilon)^2 - 1}{(1+\varepsilon)^2 + 1} = \varepsilon \left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon^3}{4} \right) + \mathcal{O}(\varepsilon^5). \quad (12)$$

Conformal invariance of capacity combined with (5) and (12) yield ($\delta = h/2$):

$$\text{Cap}(C_1) = \text{Cap}(C'_1) = \frac{\gamma}{\varepsilon} + \frac{1}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(1). \quad (13)$$

To find out more about the capacity of C'_1 we map the interior of the rectangle with vertices $0, \omega_1, \omega_1 + i\omega_2, i\omega_2$ in the complex $\overline{\mathbb{C}}_u$ plane onto the field of C'_1 endowed with a slit along the real axis connecting the plates and going through infinity (see Figure 1). The modulus $M(\Gamma)$ of the family Γ comprising vertical line segments which connect the top and the bottom side of the rectangle

equals ω_1/ω_2 . The conformal image Γ' of Γ under the mapping $u \rightarrow z(u)$ is the family of field lines of the condenser C'_1 . Obviously, the slit cannot change the modulus of Γ' since no field line can cross it due to symmetry. Conformal invariance and identity (2) give:

$$\text{Cap}(C'_1) = M(\Gamma') = M(\Gamma) = \omega_1/\omega_2. \quad (14)$$

The mapping $z(u)$ will be constructed in the lemma below. Before we proceed to formulating it let us remind the definitions of theta-functions [2, 7]:

$$\vartheta_1(w; q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin((2n+1)w) = 2q^{1/4} \sin w \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2w + q^{4n}), \quad (15)$$

$$\vartheta_2(w; q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos((2n+1)w) = 2q^{1/4} \cos w \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2w + q^{4n}), \quad (16)$$

$$\vartheta_3(w; q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nw) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1} e^{2iw})(1 + q^{2n-1} e^{-2iw}), \quad (17)$$

$$\vartheta_4(w; q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nw) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1} e^{2iw})(1 - q^{2n-1} e^{-2iw}). \quad (18)$$

The following properties of ϑ_i will prove useful in the sequel [2, 7] (we will omit q in $\vartheta_i(w; q)$ when it cannot lead to confusion):

$$\vartheta_4(w + \pi) = \vartheta_4(w) \quad (19)$$

$$\vartheta_4(w + \pi\tau) = -q^{-1} e^{-2iw} \vartheta_4(w), \quad \tau = \frac{1}{i\pi} \ln q, \quad (20)$$

$$\vartheta_4(w + \pi\tau/2; q) = i e^{-i\pi\tau/4} e^{-iy} \vartheta_1(w; q), \quad (21)$$

$$\vartheta_4(w + \pi/2; q) = \vartheta_3(w; q), \quad (22)$$

$$\vartheta_i(\overline{w}) = \overline{\vartheta_i(w)}, \quad i = 1, 3, 4, \quad (23)$$

$$\Im(\vartheta_i(ix)) = 0, \quad x \in \mathbf{R}, \quad i = 1, 3, 4, \quad (24)$$

$$\vartheta_i(-w) = \vartheta_i(w), \quad i = 3, 4, \quad (25)$$

$$\vartheta_1(-w) = -\vartheta_1(w), \quad (26)$$

$$\vartheta_4(\pi\tau/2 + \pi m + \pi n) = 0, \quad n, m \in \mathbf{Z}. \quad (27)$$

Lemma 1 *Let $\omega_1 > 0$, $\omega_2 > 0$ and $1 < R < e^{\pi\omega_2/\omega_1}$. The function*

$$z(u) = R \frac{\vartheta_4(\pi u/\omega_1 - (i \ln R)/2; q)}{\vartheta_4(\pi u/\omega_1 + (i \ln R)/2; q)}, \quad q = e^{i\pi\tau}, \quad \tau = i \frac{\omega_2}{\omega_1}, \quad (28)$$

maps the rectangle with vertices $0, \omega_1, \omega_1 + i\omega_2, i\omega_2$ in the complex $\overline{\mathbb{C}}_u$ plane conformally and univalently onto the entire $\overline{\mathbb{C}}_z$ plane with two circular slits with radii R and $1/R$ symmetric with respect to real axis and connected by the slit along the real axis going through infinity.

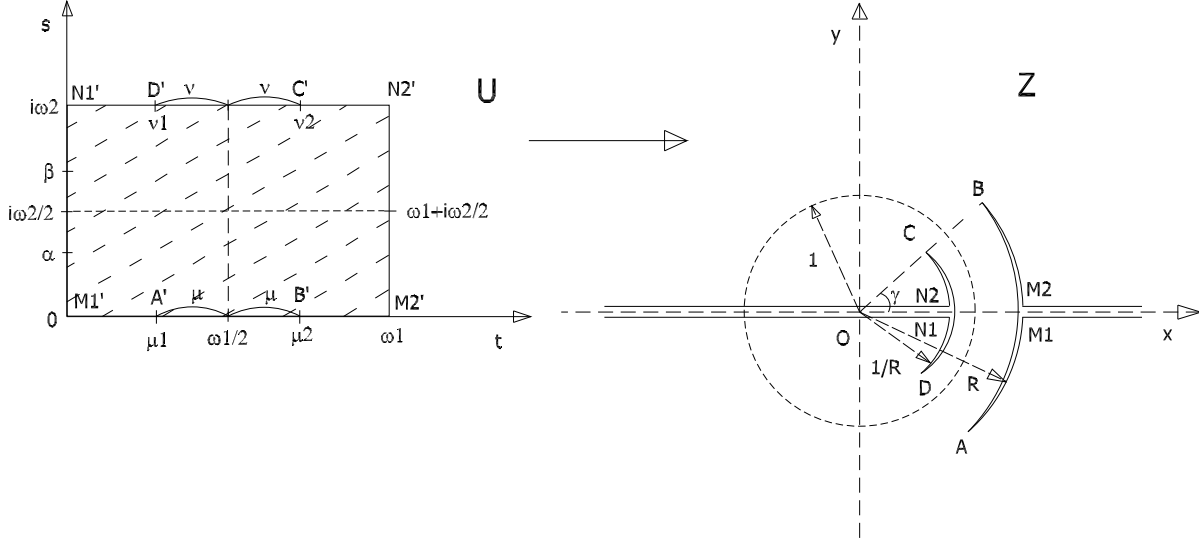


Figure 1: The mapping realized by $z(u)$

The mapping (28) was essentially constructed in [12]. For completeness we give a direct proof here.

Proof. Let u traverse the boundary of the rectangle shown on Figure 1. To trace its image $z(u)$ we first note that by (19) and (20) ($\tau = i\omega_2/\omega_1$):

$$z(u + \omega_1) = z(u), \quad (29)$$

$$z(u + \omega_2) = R \frac{\vartheta_4(\pi u/\omega_1 - (i \ln R)/2 + \pi\tau; q)}{\vartheta_4(\pi u/\omega_1 + (i \ln R)/2 + \pi\tau; q)} = \frac{-q^{-1}e^{-2i(\pi u/\omega_1 - (i \ln R)/2)}}{-q^{-1}e^{-2i(\pi u/\omega_1 + (i \ln R)/2)}} z(u) = \frac{z(u)}{R^2}. \quad (30)$$

For the bottom side (M_1', M_2') write $u = t \in [0, \omega_1]$. We employ (21) to get:

$$|z(t)| = R \left| \frac{\vartheta_4(\pi t/\omega_1 - (i \ln R)/2)}{\vartheta_4(\pi t/\omega_1 + (i \ln R)/2)} \right| = R$$

and

$$z(0) = z(\omega_1) = R \frac{\vartheta_4(-i \ln R/2)}{\vartheta_4(i \ln R/2)} = R \frac{\overline{\vartheta_4(i \ln R/2)}}{\vartheta_4(i \ln R/2)} = R$$

by (23) and (24). It follows that the interval $[0, \omega_1]$ is mapped onto a circular arc beginning and ending at the point $z = R$. This arc is symmetric with respect to real axis since for any $\lambda \in (0, \omega_1/2)$:

$$\begin{aligned} z(\omega_1/2 + \lambda) &= R \frac{\vartheta_4(\pi\lambda/\omega_1 - (i \ln R)/2) + \pi/2}{\vartheta_4(\pi\lambda/\omega_1 + (i \ln R)/2) + \pi/2} = R \frac{\vartheta_3(\pi\lambda/\omega_1 - (i \ln R)/2)}{\vartheta_3(\pi\lambda/\omega_1 + (i \ln R)/2)} = \\ &= R \frac{\overline{\vartheta_3(\pi\lambda/\omega_1 + (i \ln R)/2)}}{\vartheta_3(\pi\lambda/\omega_1 - (i \ln R)/2)} = R \frac{\overline{\vartheta_3(-\pi\lambda/\omega_1 - (i \ln R)/2)}}{\vartheta_3(-\pi\lambda/\omega_1 + (i \ln R)/2)} = \overline{z(\omega_1/2 - \lambda)} \end{aligned}$$

by (23) and (25). In particular, $z(\omega_1/2) = R$.

For the top side (N'_1, N'_2) write $u = t + i\omega_2$, $t \in [0, \omega_1]$, and by (30):

$$|z(t + i\omega_2)| = \frac{|z(t)|}{R^2} = \frac{1}{R}.$$

Previous calculations for the bottom side combined with (30) show that the side (N'_1, N'_2) is mapped onto a circular arc beginning and ending at the point $z = 1/R$ which is symmetric with respect to real axis.

When $u \in [0, i\omega_2]$ we see that $z(u)$ is real by (24). Due to periodicity (29) the same values are assumed by $z(u)$ when $u \in [\omega_1, \omega_1 + i\omega_2]$. Suppose α is the preimage of infinity, so that $z(\alpha) = \infty$, and β is preimage of origin, so that $z(\beta) = 0$. Then according to (27):

$$\frac{\pi\alpha}{\omega_1} + \frac{1}{2}i \ln R = \frac{i\pi\omega_2}{2\omega_1} \Leftrightarrow \alpha = i \left(\frac{\omega_2}{2} - \frac{\omega_1}{2\pi} \ln R \right)$$

and

$$\frac{\pi\beta}{\omega_1} - \frac{1}{2}i \ln R = \frac{i\pi\omega_2}{2\omega_1} \Leftrightarrow \beta = i \left(\frac{\omega_2}{2} + \frac{\omega_1}{2\pi} \ln R \right).$$

Hence α lies in $(0, i\omega_2/2)$ when $1 < R < e^{\pi\omega_2/\omega_1}$ and α and β are symmetric with respect to $i\omega_2/2$. For $u = t + i\omega_2/2$, $t \in [0, \omega_1]$, we get by (21):

$$\begin{aligned} z(t + i\omega_2/2) &= R \frac{\vartheta_4(\pi t/\omega_1 - (i \ln R)/2 + \pi\tau/2)}{\vartheta_4(\pi t/\omega_1 + (i \ln R)/2 + \pi\tau/2)} \\ &= R \frac{e^{i\pi\tau/4 - i(\pi\tau/\omega_1 - (i \ln R)/2)} \vartheta_1(\pi t/\omega_1 - (i \ln R)/2)}{e^{i\pi\tau/4 - i(\pi\tau/\omega_1 + (i \ln R)/2)} \vartheta_1(\pi t/\omega_1 + (i \ln R)/2)} = \frac{\vartheta_1(\pi t/\omega_1 - (i \ln R)/2)}{\vartheta_1(\pi t/\omega_1 + (i \ln R)/2)}. \end{aligned}$$

Hence by (23)

$$|z(t + i\omega_2/2)| = 1$$

and by (26)

$$z(i\omega_2/2) = z(\omega_1 + i\omega_2/2) = -1.$$

Thus the dotted line on Figure 1 connecting $i\omega_2/2$ and $\omega_1 + i\omega_2/2$ is mapped onto the unit circle in $\overline{\mathbb{C}_z}$ plane.

Finally, we see that both arcs have the same angular spread, so that the points C and B lie on the same beam. This is an obvious consequence of (30) since

$$\arg[z(t + i\omega_2)] = \arg[z(t)/R^2] = \arg[z(t)].$$

It follows that $\nu = \mu$ on Figure 1. We denote the angle between real axis and the beam OCB by γ so that

$$\gamma \equiv \max_{t \in (0, \omega_1)} \arg[z(t)]. \quad (31)$$

Above speculations are summarized the in the following chart:

u	0	μ_1	$\omega_1/2$	μ_2	ω_1	$\omega_1 + \alpha$	$\omega_1 + i\omega_2/2$	$\omega_1 + \beta$	$\omega_1 + i\omega_2$
z	R	$Re^{-i\gamma}$	R	$Re^{i\gamma}$	R	∞	-1	0	$1/R$

u	ν_2	$\omega_1/2 + i\omega_2$	ν_1	$i\omega_2$	β	$i\omega_2/2$	α
z	$e^{i\gamma}/R$	$1/R$	$e^{-i\gamma}/R$	$1/R$	0	-1	∞

This completes the proof of the lemma.

3. Equation for the capacity of C'_1 . In this section we will derive a transcendental equation for $\text{Cap}(C'_1)$ which can be viewed as a generalization of (3). We will regard γ and R (and hence $\text{Cap}(C'_1)$) as being fixed. From (14) and (28) q is also fixed and computed by

$$q = e^{-\pi/\text{cap}(C'_1)}. \quad (32)$$

We still have a free scaling factor ω_1 . Choose

$$\omega_1 = 2K(q) = \pi \vartheta_3^2(0; q) = \pi \left(1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right)^2. \quad (33)$$

Then according to (14) and (32):

$$\omega_2 = \frac{2}{\pi} K(q) \ln \frac{1}{q} = \vartheta_3^2(0; q) \ln \frac{1}{q} = \ln \frac{1}{q} \left(1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right)^2 = 2K'(q). \quad (34)$$

K and K' defined by (33) and (34) are known to be the complete elliptic integrals of the moduli

$$k^2 = \frac{\vartheta_2^4(0; q)}{\vartheta_3^4(0; q)} = 16 \left[\frac{\sum_{n=0}^{\infty} q^{(n+1/2)^2}}{1 + 2 \sum_{n=1}^{\infty} q^{n^2}} \right]^4 \quad (35)$$

and

$$k'^2 = 1 - k^2 = \frac{\vartheta_4^4(0; q)}{\vartheta_3^4(0; q)} = \left[\frac{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}}{1 + 2 \sum_{n=1}^{\infty} q^{n^2}} \right]^4, \quad (36)$$

respectively [2, §30], that is:

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K' = K(k') = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}. \quad (37)$$

Define

$$\theta_4(w; q) = \vartheta_4(\pi w/2K; q). \quad (38)$$

Then by (28):

$$z(u) = R \frac{\theta_4(u - i\alpha; q)}{\theta_4(u + i\alpha; q)}, \quad (39)$$

where we introduced

$$\alpha = \frac{1}{\pi} K \ln R > 0. \quad (40)$$

According to [2, §29]:

$$\Pi(u, i\alpha) = \frac{1}{2} \ln \frac{\theta_4(u - i\alpha; q)}{\theta_4(u + i\alpha; q)} + uZ(i\alpha) = \frac{1}{2} \ln(z(u)/R) + uZ(i\alpha), \quad (41)$$

where

$$\begin{aligned} \Pi(u, i\alpha) &= k^2 \text{sn}(i\alpha) \text{cn}(i\alpha) \text{dn}(i\alpha) \int_0^u \frac{\text{sn}^2(t) dt}{1 - k^2 \text{sn}^2(i\alpha) \text{sn}^2(t)} \\ &= k^2 b \sqrt{(1-b^2)(1-k^2b^2)} \int_0^x \frac{t^2 dt}{(1 - k^2 b^2 t^2) \sqrt{(1-t^2)(1-k^2 t^2)}} \end{aligned} \quad (42)$$

and

$$x = \operatorname{sn}(u, k), \quad b = \operatorname{sn}(i\alpha, k). \quad (43)$$

The Jacobi Z -function is defined by

$$Z(i\alpha) = \frac{\theta'_4(i\alpha; q)}{\theta_4(i\alpha; q)} = E(\operatorname{sn}(i\alpha), k) - \frac{E(k)}{K(k)} i\alpha \quad (44)$$

where

$$E(b, k) = \int_0^b \frac{\sqrt{1 - k^2 t^2} dt}{\sqrt{1 - t^2}} = \int_0^{i\alpha} [\operatorname{dn}(t, k)]^2 dt, \quad (45)$$

is the incomplete elliptic integral of the second kind. The function $\Pi(u, i\alpha)$ can be expressed in terms of incomplete elliptic integrals of the first kind

$$F(x, k) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \quad (46)$$

and of the third kind

$$\Pi(x, \nu, k) = \int_0^x \frac{dt}{(1 + \nu t^2) \sqrt{(1 - t^2)(1 - k^2 t^2)}} \quad (47)$$

by means of

$$\Pi(u, i\alpha) = -\frac{1}{b} \sqrt{(1 - b^2)(1 - k^2 b^2)} [F(x, k) - \Pi(x, -k^2 b^2, k)]. \quad (48)$$

For $u \in [0, K]$ put $\gamma(u) = \operatorname{Arg}(z(u)) \in (-\pi, \pi]$ - the principal value of argument z . Since $|z(u)| = R$

$$\ln z(u) = \ln R + i\gamma(u).$$

On the other hand from (41):

$$\ln z(u) = 2\Pi(u, i\alpha) - 2uZ(i\alpha) + \ln R.$$

Since b defined in (43) equals $i\operatorname{sn}(\alpha, k')/\operatorname{cn}(\alpha, k')$ according to [2, 7] and so is an imaginary number, we see from (44), (45) and (48) that for positive real u and α the values taken by $\Pi(u, i\alpha)$ and $Z(u, i\alpha)$ lie on the imaginary axis. Hence

$$\gamma(u) = \Im \ln z(u) = 2(\Pi(u, i\alpha) - uZ(i\alpha))/i.$$

The single extremum of $\gamma(u)$ for $u \in [0, K]$ is a minimum at $u = \mu_1$ (equal to $-\gamma$, see Figure 1). Hence we can find μ_1 by setting $d\gamma(u)/du$ to zero:

$$i \frac{d\gamma(u)}{du} = \frac{2k^2 \operatorname{sn}(i\alpha) \operatorname{cn}(i\alpha) \operatorname{dn}(i\alpha) \operatorname{sn}^2(u)}{1 - k^2 \operatorname{sn}^2(i\alpha) \operatorname{sn}^2(u)} - 2Z(i\alpha) = 0,$$

where we used (42). Finally,

$$\lambda^2(R, q) \equiv \operatorname{sn}^2(\mu_1) = \frac{Z(i\alpha)}{k^2 \operatorname{sn}(i\alpha) [\operatorname{cn}(i\alpha) \operatorname{dn}(i\alpha) + \operatorname{sn}(i\alpha) Z(i\alpha)]} \quad (49)$$

and

$$\mu_1 = F\left(\frac{\sqrt{Z(i\alpha)/\operatorname{sn}(i\alpha)}}{k \sqrt{\operatorname{cn}(i\alpha) \operatorname{dn}(i\alpha) + \operatorname{sn}(i\alpha) Z(i\alpha)}}, k\right) = F(\lambda, k). \quad (50)$$

The value of γ defined by (31) is then revealed from:

$$\gamma = 2i(\Pi(\mu_1, i\alpha) - \mu_1 Z(i\alpha)),$$

or by (48) and (43):

$$\gamma = 2i \frac{\text{cn}(i\alpha) \text{dn}(i\alpha)}{\text{sn}(i\alpha)} [\Pi(\lambda(R, q), -k^2 \text{sn}^2(i\alpha), k) - F(\lambda(R, q), k)] - 2iF(\lambda(R, q), k)Z(i\alpha). \quad (51)$$

Thus we arrive at the following statement.

Theorem 2 *Let the condenser $C'_1 = (E_0, E_1)$ be defined by (9), (10). Then its capacity $\text{Cap}(C'_1)$ satisfies transcendental equation (51) with λ , α , k , K and q defined by (49), (40), (35), (33) and (32), respectively.*

4. Proof of Theorem 1. The crucial step of the proof is to obtain an asymptotic expansion for $\text{Cap}(C'_1)$ as $R \rightarrow 1$, while γ remains fixed. After this will have been accomplished carrying out the conformal mappings inverse to those that led from C_ρ to C'_1 will complete the proof.

First we introduce the new nome q_1 which tends to zero when $q \rightarrow 1$:

$$q_1 = e^{-i\pi/\tau} = e^{\pi^2/\ln(q)} = e^{-\pi \text{Cap}(C'_1)}. \quad (52)$$

In terms of q_1 we obtain using the Jacobi transformations (and bearing in mind that $\sqrt{-i\tau} = \sqrt{\pi}[\ln(1/q_1)]^{-1/2}$ by (52)):

$$\vartheta_1(z, q) = \frac{i}{\sqrt{\pi}} [\ln(1/q_1)]^{1/2} e^{z^2 \ln(q_1)/\pi^2} \vartheta_1(iz \ln(q_1)/\pi, q_1), \quad (53)$$

$$\vartheta_2(z, q) = \frac{1}{\sqrt{\pi}} [\ln(1/q_1)]^{1/2} e^{z^2 \ln(q_1)/\pi^2} \vartheta_4(iz \ln(q_1)/\pi, q_1), \quad (54)$$

$$\vartheta_3(z, q) = \frac{1}{\sqrt{\pi}} [\ln(1/q_1)]^{1/2} e^{z^2 \ln(q_1)/\pi^2} \vartheta_3(iz \ln(q_1)/\pi, q_1), \quad (55)$$

$$\vartheta_4(z, q) = \frac{1}{\sqrt{\pi}} [\ln(1/q_1)]^{1/2} e^{z^2 \ln(q_1)/\pi^2} \vartheta_2(iz \ln(q_1)/\pi, q_1). \quad (56)$$

Denote

$$\eta = \frac{1}{2\pi} \ln(R) \ln(1/q_1) = \frac{1}{2} \ln(R) \text{Cap}(C'_1). \quad (57)$$

For the the Jacobi elliptic functions we obtain ($\alpha = K \ln(R)/\pi$):

$$\text{sn}(i\alpha, k) = \frac{1}{\sqrt{k}} \frac{\vartheta_1(i\alpha\pi/(2K), q)}{\vartheta_4(i\alpha\pi/(2K), q)} = \frac{i}{\sqrt{k}} \frac{\vartheta_1(\eta, q_1)}{\vartheta_2(\eta, q_1)}, \quad (58)$$

$$\text{cn}(i\alpha, k) = \frac{\sqrt{k'}}{\sqrt{k}} \frac{\vartheta_2(i\alpha\pi/(2K), q)}{\vartheta_4(i\alpha\pi/(2K), q)} = \frac{\sqrt{k'}}{\sqrt{k}} \frac{\vartheta_4(\eta, q_1)}{\vartheta_2(\eta, q_1)}, \quad (59)$$

$$\text{dn}(i\alpha, k) = \sqrt{k'} \frac{\vartheta_3(i\alpha\pi/(2K), q)}{\vartheta_4(i\alpha\pi/(2K), q)} = \sqrt{k'} \frac{\vartheta_3(\eta, q_1)}{\vartheta_2(\eta, q_1)} \quad (60)$$

and using (38) and (44):

$$Z(i\alpha, k) = \frac{i \ln(q_1)}{2K} \left(\frac{1}{\pi} \ln(R) + \frac{\vartheta'_2(\eta, q_1)}{\vartheta_2(\eta, q_1)} \right). \quad (61)$$

Setting $\varepsilon = R - 1$, we immediately derive from

$$\ln(1 + \varepsilon) = \varepsilon(1 - \varepsilon/2 + \varepsilon^2/3) + \mathcal{O}(\varepsilon^4)$$

(13) and (57):

$$\eta = \frac{\gamma}{2} + \frac{\varepsilon}{2\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon). \quad (62)$$

For theta functions we get from (15)-(17) for $q_1 \rightarrow 0$:

$$\vartheta_1(\eta, q_1) = 2q_1^{1/4}(\sin(\eta) + \mathcal{O}(q_1^2)), \quad (63)$$

$$\vartheta_2(\eta, q_1) = 2q_1^{1/4}(\cos(\eta) + \mathcal{O}(q_1^2)) \quad (64)$$

$$\vartheta_3(\eta, q_1) = 1 + 2q_1 \cos(2\eta) + \mathcal{O}(q_1^4), \quad (65)$$

$$\vartheta_4(\eta, q_1) = 1 - 2q_1 \cos(2\eta) + \mathcal{O}(q_1^4), \quad (66)$$

$$\vartheta'_2(\eta, q_1) = 2q_1^{1/4}(-\sin(\eta) + \mathcal{O}(q_1^2)). \quad (67)$$

For the complete elliptic integral K we can write according to (33) and (55) as $q_1 \rightarrow 0$:

$$K(q_1) = \frac{\pi}{2} \vartheta_3^2(0, q) = \frac{1}{2} \ln(1/q_1) \vartheta_3^2(0, q_1) = \frac{1}{2} \ln \frac{1}{q_1} \left(1 + 2 \sum_{n=1}^{\infty} q_1^{n^2} \right)^2 = \frac{1}{2} \ln(1/q_1) (1 + 4q_1 + \mathcal{O}(q_1^2)). \quad (68)$$

For moduli k and $k' = \sqrt{1 - k^2}$ we get according to (35), (36), (54), (55) and (64)-(66):

$$\begin{aligned} k &= \frac{\vartheta_4^2(0, q_1)}{\vartheta_3^2(0, q_1)} = \frac{(1 - 2q_1 + \mathcal{O}(q_1^4))^2}{(1 + 2q_1 + \mathcal{O}(q_1^4))^2} = \frac{1 - 4q_1 + 4q_1^2 + \mathcal{O}(q_1^4)}{1 + 4q_1 + 4q_1^2 + \mathcal{O}(q_1^4)} = \\ &= (1 - 4q_1 + 4q_1^2 + \mathcal{O}(q_1^4))(1 - 4q_1 - 4q_1^2 + 16(q_1 + q_1^2 + \mathcal{O}(q_1^4))^2 - 64(q_1 + \mathcal{O}(q_1^2))^3 + \mathcal{O}(q_1^4)) = \\ &= (1 - 4q_1 + 4q_1^2 + \mathcal{O}(q_1^4))(1 - 4q_1 + 12q_1^2 - 32q_1^3 + \mathcal{O}(q_1^4)) = 1 - 8q_1 + 32q_1^2 - 96q_1^3 + \mathcal{O}(q_1^4). \end{aligned} \quad (69)$$

Hence for $q_1 \rightarrow 0$:

$$k^2 = 1 - 16q_1 + 128q_1^2 - 704q_1^3 + \mathcal{O}(q_1^4), \quad (70)$$

$$\sqrt{k} = 1 - 4q_1 + 8q_1^2 - 16q_1^3 + \mathcal{O}(q_1^4), \quad (71)$$

and

$$1/\sqrt{k} = 1 + 4q_1 + 8q_1^2 + 16q_1^3 + \mathcal{O}(q_1^4), \quad (72)$$

where the expansions (valid for $z \rightarrow 0$):

$$(1+z)^r = 1 + rz + \binom{r}{2} z^2 + \binom{r}{3} z^3 + \binom{r}{4} z^4 + \mathcal{O}(z^5), \quad \frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \mathcal{O}(z^5). \quad (73)$$

have been used.

In a similar fashion:

$$k' = \frac{\vartheta_2^2(0, q_1)}{\vartheta_3^2(0, q_1)} = 4q_1^{1/2}(1 - 4q_1 + 14q_1^2 - 40q_1^3 + \mathcal{O}(q_1^4)), \quad (74)$$

$$\sqrt{k'} = 2q_1^{1/4}(1 - 2q_1 + 5q_1^2 - 10q_1^3 + \mathcal{O}(q_1^4)) \quad (75)$$

and

$$\sqrt{k'/k} = 2q_1^{1/4}(1 + 2q_1 + 5q_1^2 + 10q_1^3 + \mathcal{O}(q_1^4)). \quad (76)$$

Then from (58)-(61) and (63)-(68):

$$\operatorname{sn}(i\alpha, k) = i \tan(\eta) + 4iq_1 \tan(\eta) + \mathcal{O}(q_1^2), \quad (77)$$

$$-k^2 \operatorname{sn}^2(i\alpha, k) = \tan^2(\eta) - 8q_1 \tan^2(\eta) + \mathcal{O}(q_1^2), \quad (78)$$

$$\operatorname{cn}(i\alpha, k) = \frac{1}{\cos(\eta)} + 4q_1 \frac{\sin^2(\eta)}{\cos(\eta)} + \mathcal{O}(q_1^2), \quad (79)$$

$$\operatorname{dn}(i\alpha, k) = \frac{1}{\cos(\eta)} - 4q_1 \frac{\sin^2(\eta)}{\cos(\eta)} + \mathcal{O}(q_1^2), \quad (80)$$

$$Z(i\alpha, k) = i \tan(\eta) - \frac{2i\eta}{\ln(1/q_1)} - 4iq_1 \tan(\eta) + \frac{8iq_1\eta}{\ln(1/q_1)} + \mathcal{O}(q_1^2), \quad (81)$$

$$2i \frac{\operatorname{cn}(i\alpha) \operatorname{dn}(i\alpha)}{\operatorname{sn}(i\alpha)} = \frac{2}{\sin(\eta) \cos(\eta)} - \frac{8q_1}{\sin(\eta) \cos(\eta)} + \mathcal{O}(q_1^2). \quad (82)$$

Introduce the notation

$$x = [\ln(1/q_1)]^{-1} = \frac{1}{\pi \operatorname{Cap}(C'_1)} \rightarrow 0 \quad \text{as } q_1 \rightarrow 0. \quad (83)$$

Using this notation and (49):

$$\begin{aligned} \lambda^2(R, q_1) &= \frac{(i \tan(\eta) - 2i\eta x + \mathcal{O}(q_1)) / [(1 + \mathcal{O}(q_1))(i \tan(\eta) + \mathcal{O}(q_1))]}{\left(\frac{1}{\cos(\eta)} + \mathcal{O}(q_1)\right) \left(\frac{1}{\cos(\eta)} + \mathcal{O}(q_1)\right) + (i \tan(\eta) + \mathcal{O}(q_1))(i \tan(\eta) - 2i\eta x + \mathcal{O}(q_1))} \\ &= \frac{1 - 2\eta x / \tan \eta + \mathcal{O}(q_1)}{1 / \cos^2 \eta - \tan^2 \eta + 2\eta x \tan \eta + \mathcal{O}(q_1)} \\ &= \frac{1 - 2\eta x / \tan \eta}{1 + 2\eta x \tan \eta} + \mathcal{O}(q_1) = 1 - \frac{2\eta x}{\sin \eta (\cos \eta + 2\eta x \sin \eta)} + \mathcal{O}(q_1). \end{aligned} \quad (84)$$

By (73) this leads to:

$$\begin{aligned} \lambda(R, q_1) &= 1 + \sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} \frac{(2\eta x)^k}{\sin^k \eta \cos^k \eta (1 + 2\eta x \tan \eta)^k} + \mathcal{O}(q_1) \\ &= \sum_{k=0}^{\infty} \binom{1/2}{k} \frac{(-2\eta x \tan \eta)^k}{\sin^{2k} \eta} \sum_{m=0}^{\infty} \binom{k+m-1}{m} (-2\eta x \tan \eta)^m + \mathcal{O}(q_1) \\ &= \sum_{n=0}^{\infty} (-2\eta x \tan \eta)^n \sum_{k=0}^n \binom{1/2}{k} \binom{n-1}{n-k} \sin^{-2k} \eta + \mathcal{O}(q_1), \end{aligned}$$

where $\binom{n-1}{n} = 0$ when $n > 0$ and $\binom{n-1}{n} = 1$, when $n = 0$ (which is standard and very convenient convention). From this:

$$\begin{aligned} \sigma \equiv 1 - \lambda(R, q_1) &= - \sum_{n=1}^{\infty} (-2\eta x \tan \eta)^n \sum_{k=1}^n \binom{1/2}{k} \binom{n-1}{n-k} \sin^{-2k} \eta + \mathcal{O}(q_1) \\ &= \frac{\eta x \tan \eta}{\sin^2 \eta} (1 - \beta(x, \eta)) + \mathcal{O}(q_1), \end{aligned} \quad (85)$$

where

$$\beta(x, \eta) = 4\eta x \tan \eta \sum_{n=0}^{\infty} (-1)^n (2\eta x \tan \eta)^n \sum_{k=0}^{n+1} \binom{1/2}{k+1} \binom{n+1}{n-k+1} \sin^{-2k} \eta = 2\eta x \tan \eta \left(1 - \frac{1}{4 \sin^2 \eta}\right)$$

$$\begin{aligned}
& -(2\eta x \tan \eta)^2 \left(1 - \frac{1}{2\sin^2 \eta} + \frac{1}{8\sin^4 \eta}\right) + (2\eta x \tan \eta)^3 \left(1 - \frac{3}{4\sin^2 \eta} + \frac{3}{8\sin^4 \eta} - \frac{5}{64\sin^6 \eta}\right) \\
& -(2\eta x \tan \eta)^4 \left(1 - \frac{1}{\sin^2 \eta} + \frac{3}{4\sin^4 \eta} - \frac{5}{16\sin^6 \eta} + \frac{7}{128\sin^8 \eta}\right) + \mathcal{O}(x^5). \tag{86}
\end{aligned}$$

Now denote:

$$\nu \equiv -k^2 \text{sn}^2(i\alpha, k) = \tan^2(\eta)(1 - 8q_1 + \mathcal{O}(q_1^2)) > 0 \quad \text{when } q_1 \text{ is small,} \tag{87}$$

$$\sqrt{\nu} = \tan(\eta)(1 - 4q_1 + \mathcal{O}(q_1^2)). \tag{88}$$

We have by (69) and (85):

$$(1 - k)/\sigma = \mathcal{O}(q_1 \ln(1/q_1)) \rightarrow 0 \text{ as } q_1 \rightarrow 0.$$

It follows from the results of [9] that for

$$k \rightarrow 1, \quad \sigma \rightarrow 0 \quad \text{and} \quad (1 - k)/\sigma \rightarrow 0 \tag{89}$$

$\Pi - F$ has the asymptotic approximation

$$\Pi(1 - \sigma, \nu, k) - F(1 - \sigma, k) = \frac{-\nu}{2(1 + \nu)} \ln \frac{2 - \sigma}{\sigma} + \frac{\sqrt{\nu} \arctan((1 - \sigma)\sqrt{\nu})}{1 + \nu} + \mathcal{O}((1 - k)/\sigma). \tag{90}$$

The Taylor expansion for $\arctan(\tan \eta(1 - \epsilon))$ for small ϵ is given by

$$\arctan(\tan \eta(1 - \epsilon)) = \eta + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n} \arctan(x)|_{x=\tan \eta} (-\epsilon \tan \eta)^n.$$

Simple manipulations reveal

$$\frac{d^n}{dx^n} \arctan(x)|_{x=\tan \eta} = -(n - 1)! \cos^{2n} \eta \sum_{m=0}^{[(n-1)/2]} (-1)^{n-m} \binom{n}{2m+1} \tan^{n-2m-1} \eta.$$

Hence

$$\begin{aligned}
\arctan(\tan \eta(1 - \epsilon)) &= \eta - \sum_{n=1}^{\infty} \frac{\epsilon^n}{n} \sin^n \eta \sum_{m=0}^{[(n-1)/2]} (-1)^m \binom{n}{2m+1} \sin^{n-2m-1} \eta \cos^{2m+1} \eta \\
&= \eta - \epsilon \cos \eta \sin \eta - \epsilon^2 \cos \eta \sin^3 \eta - \frac{\epsilon^3}{3} \cos \eta \sin^3 \eta (4 \sin^2 \eta - 1) - \epsilon^4 \cos \eta \sin^5 \eta (2 \sin^2 \eta - 1) + \mathcal{O}(\epsilon^5).
\end{aligned}$$

The Taylor expansion for $\ln(2 - \sigma)$ is:

$$\ln(2 - \sigma) = \ln(2) - \sum_{m=1}^{\infty} \frac{\sigma^m}{m2^m}.$$

Using these Taylor expansions, (87) and (88) expansion (90) is transformed into

$$\begin{aligned}
\Pi(\lambda, \nu, k) - F(\lambda, k) &= \frac{-\tan^2 \eta (\ln(1/\sigma) + \ln(2))}{2(1 + \tan^2 \eta + \mathcal{O}(q_1))} + \frac{(\tan \eta + \mathcal{O}(q_1)) \arctan((1 - \sigma)(\tan \eta + \mathcal{O}(q_1)))}{1 + \tan^2 \eta + \mathcal{O}(q_1)} \\
&+ \frac{\tan^2 \eta + \mathcal{O}(q_1)}{1 + \tan^2 \eta + \mathcal{O}(q_1)} \left(\sum_{m=1}^{\infty} \frac{\sigma^m}{m2^{m+1}} \right) + \mathcal{O}((1 - k)/\sigma) = -\frac{1}{2} \sin^2 \eta \ln(1/\sigma) - \frac{1}{2} \sin^2 \eta \ln 2
\end{aligned}$$

$$\begin{aligned}
& + \frac{\tan \eta \arctan(\tan \eta [1 - \sigma + \mathcal{O}(q_1)])}{1 + \tan^2 \eta} + \sin^2 \eta \left(\sum_{m=1}^{\infty} \frac{\sigma^m}{m 2^{m+1}} \right) + \mathcal{O}(q_1 \ln(1/q_1)) = -\frac{1}{2} \sin^2 \eta \ln(1/\sigma) \\
& - \frac{\sin^2 \eta \ln 2}{2} + \frac{\eta \sin(2\eta)}{2} + \sum_{n=1}^{\infty} \frac{\sigma^n}{n} \left[\frac{\sin^2 \eta}{2^{n+1}} - \sum_{m=0}^{[(n-1)/2]} (-1)^m \binom{n}{2m+1} \cos^{2m+2} \eta \sin^{2n-2m} \eta \right] + \mathcal{O}\left(q_1 \ln \frac{1}{q_1}\right) \\
& = -\frac{1}{2} \sin^2 \eta \ln \frac{1}{\sigma} - \frac{1}{2} \sin^2 \eta \ln(2) + \frac{1}{2} \eta \sin(2\eta) + \sigma \sin^2 \eta (1/4 - \cos^2 \eta) + \sigma^2 \sin^2 \eta (1/16 - \cos^2 \eta \sin^2 \eta) \\
& + \sigma^3 \sin^2 \eta (1/48 - \cos^2 \eta \sin^4 \eta + \cos^4 \eta \sin^2 \eta / 3) + \sigma^4 \sin^2 \eta (1/128 - \cos^2 \eta \sin^6 \eta + \cos^4 \eta \sin^4 \eta) + \mathcal{O}(\sigma^5).
\end{aligned}$$

This and (82) lead to

$$\begin{aligned}
2i \frac{\text{cn}(i\alpha) \text{dn}(i\alpha)}{\text{sn}(i\alpha)} (\Pi(1 - \sigma, \nu, k) - F(1 - \sigma, k)) &= -\tan \eta \ln \frac{1}{\sigma} - \tan \eta \ln(2) + 2\eta + 2\sigma \tan \eta (1/4 - \cos^2 \eta) \\
&+ 2\sigma^2 \tan \eta (1/16 - \cos^2 \eta \sin^2 \eta) + 2\sigma^3 \tan \eta (1/48 - \cos^2 \eta \sin^4 \eta + \cos^4 \eta \sin^2 \eta / 3) \\
&+ 2\sigma^4 \tan \eta (1/128 - \cos^2 \eta \sin^6 \eta + \cos^4 \eta \sin^4 \eta) + \mathcal{O}(\sigma^5). \quad (91)
\end{aligned}$$

Substituting (85) for σ and denoting for brevity

$$z = 2\eta x \tan \eta, \quad s = \frac{1}{\sin^2 \eta},$$

we obtain:

$$\begin{aligned}
\ln \frac{1}{\sigma} &= \ln \frac{1}{x} + \ln \frac{\sin 2\eta}{2\eta} + \sum_{k=1}^{\infty} \frac{\beta^k(x, \eta)}{k} = \\
&= \ln \frac{1}{x} + \ln \frac{\sin 2\eta}{2\eta} + z(1 - s/4) + z^2(-1/2 + s/4 - 3s^2/32) + z^3(1/3 - s/4 + 3s^2/16 - 5s^3/96) \\
&+ z^4(-1/4 + s/4 - 9s^2/32 + 5s^3/32 - 35s^4/1024) + \mathcal{O}(x^5).
\end{aligned}$$

For the powers of σ we compute up to $\mathcal{O}(x^5)$:

$$\begin{aligned}
\sigma &= zs/2 - z^2(s/2 - s^2/8) + z^3(s/2 - s^2/4 + s^3/16) - z^4(s/2 - 3s^2/8 + 3s^3/16 - 5s^4/128) + \mathcal{O}(x^5), \\
\sigma^2 &= z^2 s^2/4 - z^3(s^2/2 - s^3/8) + z^4(3s^2/4 - 3s^3/8 + 5s^3/64) + \mathcal{O}(x^5), \\
\sigma^3 &= z^3 s^3/8 - z^4(3s^3/8 - 3s^4/32) + \mathcal{O}(x^5), \\
\sigma^4 &= z^4 s^4/16 + \mathcal{O}(x^5).
\end{aligned}$$

Hence (91) transforms into

$$\begin{aligned}
2i \frac{\text{cn}(i\alpha) \text{dn}(i\alpha)}{\text{sn}(i\alpha)} (\Pi(1 - \sigma, \nu, k) - F(1 - \sigma, k)) &= -\tan \eta \ln \frac{1}{x} + 2\eta - \tan \eta \ln \frac{\sin 2\eta}{\eta} \\
&- \frac{\eta x}{\cos^2 \eta} - \frac{\eta^2 x^2 (1 - 4 \sin^2 \eta)}{4 \cos^3 \eta \sin \eta} - \frac{\eta^3 x^3 (1 - 4 \sin^2 \eta + 8 \sin^4 \eta)}{6 \cos^4 \eta \sin^2 \eta} \\
&- \frac{\eta^4 x^4 (10 - 109 \sin^2 \eta + 208 \sin^4 \eta - 144 \sin^6 \eta)}{32 \sin^3 \eta \cos^5 \eta} + \mathcal{O}(x^5).
\end{aligned}$$

For $2iF(\lambda, k)Z(i\alpha, k)$ from (81) and

$$F(1 - \sigma, k) = \frac{1}{2} \ln(1/\sigma) + \frac{1}{2} \ln(2) - \sum_{m=1}^{\infty} \frac{\sigma^m}{m 2^{m+1}} + \mathcal{O}(q_1 \ln(q_1))$$

we get representation

$$\begin{aligned}
2iF(\lambda, k)Z(i\alpha, k) = & -\tan \eta \ln(1/x) - \tan \eta \ln \frac{\sin(2\eta)}{\eta} + 2\eta x \ln(1/x) + \eta x \left(\frac{\cos(2\eta)}{\cos^2 \eta} + 2 \ln \frac{\sin(2\eta)}{\eta} \right) \\
& - \frac{\eta^2 x^2 (5 - 16 \sin^2 \eta + 8 \sin^4 \eta)}{4 \cos^3 \eta \sin \eta} - \frac{\eta^3 x^3 (4 - 15 \sin^2 \eta + 24 \sin^4 \eta - 8 \sin^6 \eta)}{6 \cos^4 \eta \sin^2 \eta} \\
& - \frac{\eta^4 x^4 (55 - 256 \sin^2 \eta + 480 \sin^4 \eta - 512 \sin^6 \eta + 128 \sin^8 \eta)}{96 \sin^3 \eta \cos^5 \eta} + \mathcal{O}(x^5).
\end{aligned}$$

Substituting last two expansions into (51) we finally arrive at the following formula:

$$\begin{aligned}
\gamma = & 2\eta - 2\eta x \ln(1/x) - 2\eta x \left(1 + \ln \frac{\sin(2\eta)}{\eta} \right) + 2\eta^2 x^2 \cot(2\eta) \\
& + 2\eta^3 x^3 \frac{3 - 2 \sin^2(2\eta)}{3 \sin^2(2\eta)} + \eta^4 x^4 \frac{(25 + 96 \sin^2 \eta - 48 \sin^4 \eta - 128 \sin^6 \eta)}{12 \sin^3(2\eta)} + \mathcal{O}(x^5). \quad (92)
\end{aligned}$$

Now denote $y = \text{Cap}(C'_1)$ and recall the rough approximations (13) and (62). These asymptotic approximations and equation (92) are sufficient ingredients to start using the bootstrapping technique for identifying further asymptotic terms in (13). Substituting (13) and (62) into (92) we can rewrite the latter as

$$\gamma = \ln(1 + \varepsilon) \left[y - \frac{1}{\pi} \ln(\pi) - \frac{1}{\pi} \ln y - \frac{1}{\pi} \left(1 + \ln \frac{2 \sin(\gamma + \frac{\varepsilon}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon))}{\gamma + \frac{\varepsilon}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)} \right) \right] + \mathcal{O}(\varepsilon^2). \quad (93)$$

The following Taylor expansions hold true:

$$\begin{aligned}
\ln y = & \ln \left[\frac{\gamma}{\varepsilon} \left(1 + \frac{\varepsilon}{\gamma \pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon) \right) \right] = \ln \frac{\gamma}{\varepsilon} + \frac{\varepsilon}{\gamma \pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon), \\
\frac{\sin(\gamma + \frac{\varepsilon}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon))}{\gamma + \frac{\varepsilon}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)} = & \frac{\sin \gamma}{\gamma} + \frac{\varepsilon}{\pi} \left(\frac{\cos \gamma}{\gamma} - \frac{\sin \gamma}{\gamma^2} \right) \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon), \\
\ln \frac{2 \sin(\gamma + \frac{\varepsilon}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon))}{\gamma + \frac{\varepsilon}{\pi} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)} = & \ln \left[\frac{2 \sin \gamma}{\gamma} \left(1 + \frac{\varepsilon \gamma}{\pi \sin \gamma} \left(\frac{\cos \gamma}{\gamma} - \frac{\sin \gamma}{\gamma^2} \right) \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon) \right) \right] = \\
= & \ln \frac{2 \sin \gamma}{\gamma} + \frac{\varepsilon}{\pi} \left(\cot \gamma - \frac{1}{\gamma} \right) \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon).
\end{aligned}$$

Substituting these expansions into (93) yields:

$$\frac{\gamma}{\varepsilon} = \left(1 - \frac{\varepsilon}{2} \right) \left(y - \frac{1}{\pi} \ln \frac{1}{\varepsilon} - \frac{1}{\pi} (1 + \ln(2\pi \sin \gamma)) - \frac{\cot \gamma}{\pi^2} \varepsilon \ln \frac{1}{\varepsilon} \right) + \mathcal{O}(\varepsilon).$$

Substituting once more $\gamma + \varepsilon \ln(1/\varepsilon)/\pi + \mathcal{O}(\varepsilon)$ for $y\varepsilon$ results in identification of the next asymptotic terms:

$$y = \frac{\gamma}{\varepsilon} + \frac{1}{\pi} \ln \frac{1}{\varepsilon} + \frac{\gamma}{2} + \frac{1}{\pi} + \frac{1}{\pi} \ln(2\pi \sin \gamma) + \frac{\cot \gamma}{\pi^2} \varepsilon \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon) \quad (94)$$

and

$$\eta = \frac{\gamma}{2} + \frac{\varepsilon}{2\pi} \ln \frac{1}{\varepsilon} + \frac{\varepsilon}{2\pi} (1 + \ln(2\pi \sin \gamma)) + \frac{\varepsilon^2}{2\pi} \left(\frac{1}{\pi} \cot \gamma - \frac{1}{2} \right) \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^2). \quad (95)$$

The next step of bootstrapping is to substitute these approximations into (92). Using the expansions (as $x \rightarrow 0$):

$$\begin{aligned}
\ln(1 + x) = & x(1 - x/2 + x^2/3) + \mathcal{O}(x^4), \\
\frac{\sin(\gamma + x)}{\gamma + x} = & \frac{\sin \gamma}{\gamma} + \frac{x}{\gamma} \left(\cos \gamma - \frac{\sin \gamma}{\gamma} \right) - \frac{x^2}{\gamma^2} \left(\frac{\gamma}{2} \sin \gamma + \cos \gamma - \frac{\sin \gamma}{\gamma} \right) + \mathcal{O}(x^3), \\
\cot(\gamma + x) = & \cot \gamma - \frac{x}{\sin^2 \gamma} + \mathcal{O}(x^2)
\end{aligned}$$

and omitting some tedious calculations we arrive at:

$$y = \frac{\gamma}{\varepsilon} + \frac{1}{\pi} \ln \frac{1}{\varepsilon} + \frac{\gamma}{2} + \frac{1}{\pi} (1 + \ln(2\pi \sin \gamma)) + \frac{\cot \gamma}{\pi^2} \varepsilon \ln \frac{1}{\varepsilon} + \varepsilon \left[\frac{\gamma}{4} + \frac{1}{2\pi} + \frac{\cot \gamma}{\pi^2} \left(\frac{1}{2} + \ln(2\pi \sin \gamma) \right) \right] - \frac{\varepsilon^2 \ln^2(1/\varepsilon)}{2\pi^3 \sin^2 \gamma} - \frac{\varepsilon^2 \ln(1/\varepsilon)}{2\pi^3} \left(1 + \pi \cot \gamma - \cot^2 \gamma + \frac{2 \ln(2\pi \sin \gamma)}{\sin^2 \gamma} \right) + \mathcal{O}(\varepsilon^2). \quad (96)$$

To derive an asymptotic expansion for the capacity of the condenser C_1 (see section 2) from this formula recall that $\text{Cap}(C_1) = \text{Cap}(C'_1)$ and substitute (11) for ε . Simple calculations lead to the expression

$$\text{Cap}(C_1) = \frac{\gamma}{\delta} + \frac{1}{\pi} \ln \frac{1}{\delta} + \frac{1}{\pi} (1 + \ln(2\pi \sin \gamma)) + \frac{\cot \gamma}{\pi^2} \delta \ln \frac{1}{\delta} + \frac{\delta \cot \gamma}{\pi^2} \left(\frac{1}{2} + \ln(2\pi \sin \gamma) \right) - \frac{\delta^2 \ln^2(1/\delta)}{2\pi^3 \sin^2 \gamma} - \frac{\delta^2 \ln(1/\delta)}{2\pi^3 \sin^2 \gamma} (2 \ln(2\pi \sin \gamma) - \cos(2\gamma)) + \mathcal{O}(\delta^2). \quad (97)$$

The final step that gives formula (8) from Theorem 1 is to carry out the conformal mapping $z \rightarrow \rho z$. Under this mapping the condenser C_1 transforms into the condenser C_ρ with $L = 2\rho\gamma$ and $h = 2\delta\rho$. Thus substituting $\gamma = L/(2\rho)$ and $\delta = h/(2\rho)$ into (97) brings the desired result.

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